Journal of Geometry and Physics 9 (1992) 45–73
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Abstract Jacobi and Poisson structures. Quantization and star-products *

J. Grabowski
Institute of Mathematics, Warsaw University, ul. Banacha 2, 00-913 Warsaw 59, Poland

Received 4 April 1991
(Revised 3 September 1991)

We develop a purely algebraic approach to Poisson and Jacobi structures on differentiable manifolds and to the corresponding star-products. We present detailed algebraic foundations for the study and, following some ideas of Drinfel'd, a constructive formula for a star-product, which can be applied in some degenerate cases as well.

Keywords: *Poisson structure, star-product, Lie algebra, associative algebra, universal enveloping algebra, differential operator*
1991 MSC: *17B65, 16A24, 58F05.*

1. Introduction

The study of deformations of algebraic structures in the sense of Gerstenhaber [13] encouraged Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3] to view quantum mechanics as a theory of functions on phase spaces with deformed products and Poisson brackets. They suggested "that quantization be understood as a deformation of the structure of the algebra of classical observables, rather than a radical change in the nature of the observables", and were able to calculate the spectrum of the hydrogen atom in this framework. This suggests also the possibility of developing new methods for quantum theories, especially quantum field theories.

Such an approach is not new. Since the Poincaré group is in a sense a deformation of the group of Galilean transformations, the theory of relativity can be viewed as a deformation of nonrelativistic physics. We obtain the nonrelativistic case by passing to a certain limit of a deformation parameter (velocity of light).

The basic mathematical structures of classical mechanics are the algebra $V=C^\infty(N)$ of smooth functions on the phase space N under ordinary multiplication and the Lie structure on V induced by the Poisson bracket defined by the

* Partially supported by the Alexander von Humboldt-Stiftung and by CPBP-127.

symplectic form. The appropriate deformations of the associative algebra structure on V are called star-products. The first star-product appeared as the inverse Weyl transform of the product of operators (Moyal [24]). It was rediscovered by Vey [28], who considered also deformations of the Poisson bracket.

The existence of star-products for symplectic structures was proved by Neroslavsky and Vlassov [25] for manifolds with a vanishing third De Rham cohomology group and finally by De Wilde and Lecomte [9] in all generality. The method consists of constructing the deformation step by step and requires deep tools from differential geometry, since one encounters obstructions in Hochschild and Chevalley cohomology. Unfortunately, the method is not constructive in the sense that we can write the star-product explicitly. On the other hand, one wants to consider more general Poisson brackets than those obtained from symplectic forms. For example, Dirac, in the course of his research on degenerate Lagrange systems, suggested that the classical Poisson bracket should be replaced by a different bracket, which is now known as the Dirac bracket. It is a particular case of what is now called a Poisson structure (cf. ref. [22]) or, more generally, a Jacobi structure in the sense of Lichnerowicz [23] or a local Lie algebra structure in the sense of Kirillov [18] (see also ref. [15]). These notions are of a geometrical nature and their precise definitions make use of the so called Nijenhuis–Schouten bracket of antisymmetric contravariant tensors on a manifold.

The abstract algebraic concept of a Poisson structure first appeared in papers by Krasil'shchik and Vinogradov [21] as "canonical algebra" and by Guillemin and Sternberg [16]. It was studied and developed by Braconnier [6], Bhaskara and Viswanath [4] and Krasil'shchik [19,20] in the graded algebra version as well. Also Atkin and the present author in refs. [1,2,14] used this concept for proving theorems concerning homomorphisms of Lie algebras associated with a symplectic form.

It is a common trend in differential geometry to algebraize notions and formulas as much as reasonable in order to work coordinate free. The well-known example of such a procedure is to understand vector fields as derivations of the algebra $C^\infty(N)$ of smooth functions on a manifold, differential forms as antisymmetric $C^\infty(N)$ multilinear mappings, etc.

The Nijenhuis–Schouten bracket is in this language a particular case of the Nijenhuis–Richardson bracket for antisymmetric multilinear mappings over a vector space [26] and one can see, comparing the clear algebraic form of the Nijenhuis–Richardson bracket with the mysterious form of the Nijenhuis–Schouten bracket expressed in local coordinates, how powerful such an algebraization can be. It simplifies and clarifies calculations which on the level of local coordinates not only may be complicated but also hide the essence of the problem.

Our aim is therefore to develop a purely algebraic approach to Poisson or Jacobi structures and star-products. The object under consideration will be an associative commutative algebra V with unit over a field of characteristic zero. We introduce an algebraic notion of linear differential operators on V in the spirit of Vinogradov (see, e.g., refs. [21,29]) and define a Jacobi structure to be a Lie algebra structure on V given by a differential operator.

This approach has many advantages. First, we can take for V not only the algebra $C^\infty(N)$, but also the algebra of polynomial, real analytic, holomorphic, etc., functions. Secondly, we have a proper language for such geometrical structures not only on manifolds but also on more general and important spaces such as spaces of orbits or leaves of generalized foliations, where the local coordinate approach is mostly inappropriate. And finally, we could deal with Poisson structures on infinite-dimensional manifolds, which gives the possibility of discovering new quantization schemes for field theories.

We start in section 2 with an algebraic formalism developed by De Wilde and Lecomte in ref. [10] concerning a graded Lie algebra structure on the space $M(V)$ of multilinear mappings over a vector space V, Hochschild and Chevalley cohomology, and deformations. Assuming that V is an associate commutative algebra, we introduce in section 3 the notion of a linear differential operator on V and some additional structures on $M(V)$.

In section 4 we define a Jacobi algebra. We show that, under some reasonable assumptions, the Jacobi structure must be given by a differential operator of order 1, and we give a precise description of this operator, which can be viewed as a generalization of the algebraic part of Kirillov's results [18]. We define a star-product and prove a formula showing that the universal enveloping algebra of a Lie algebra is obtained from a star-product for the canonical Poisson structure on the dual to the Lie algebra.

Section 5 is devoted to a graded Lie algebra structure, cohomology, and star-products in the tensor algebra of a universal enveloping algebra, which are compatible with these notions for multidifferential operators. This can be regarded as a translation of the problem of existence of star-products to a formal level. Using the graded Lie algebra structure we define Poisson elements and point out a connection with the classical Yang–Baxter equations and some results of Drinfel'd [12].

A theorem there, concerning cohomology, shows what was hidden behind the proofs of Vey [28] and Cahen, De Wilde and Gutt [7] concerning the description of the local Hochschild cohomology of the algebras of smooth functions on a manifold.

We use the results of section 5 in the last section, where we prove the existence of star-products of order 3 in general, and star-products of infinite order in a special case. In particular, for some Poisson structure we get explicitly defined star-products, which can be viewed as generalizations of Moyal products. Note that the method works for degenerate Poisson structures as well. We end with some examples of Poisson structures and corresponding star-products. It is worth noting that, since quantum groups can be understood as deformations of Hopf algebras of functions on topological groups, the associative part of these deformations can probably be written as a star-product for the Lie–Poisson structure on the group which is the infinitesimal part of this deformation (quantum group). Therefore the star-product associated to a quantum group will give an interesting approach to the noncommutative differential geometry in the spirit of Connes [8] and, we believe, will open new perspectives to understand quantum group structures. We shall discuss these questions elsewhere.

2. Preliminaries

In this section we shall mainly use definitions and results of De Wilde and Lecomte [10], which seem to form the proper algebraic setting of the problem.

Let V be a vector space over a field k of characteristic 0. We denote by $M^p(V)$ the space of all $(p+1)$-linear maps $A: V^{p+1} \to V$ $(p \geqslant 0)$, $M^{-1}(V) = V$, and we set

$$M(V) = \bigoplus_{p \geqslant -1} M^p(V) .$$

For the graded vector space $M(V)$ define

$$i: M(V)^2 \to M(V)$$

by $i(B)A = 0$ if $A \in M^{-1}(V)$ and

$$i(B)A(x_0, ..., x_{a+b}) = \sum_{k=0}^{a} (-1)^{kb} A(x_0, ..., x_{k-1}, B(x_k, ..., x_{k+b}), x_{k+b+1}, ..., x_{a+b}) ,$$

if $A \in M^a(V)$ $(a \geqslant 0)$ and $B \in M^b(V)$. Define now $\Delta: M(V)^2 \to M(V)$ by

$$A \Delta B = i(B)A + (-1)^{ab+1} i(A)B, \quad A \in M^a(V), \quad B \in M^b(V) .$$

The "bracket" Δ in $M(V)$ is an extension of the usual commutator bracket in $M^0(V)$.

For the graded subspace $\mathscr{A}(V) = \oplus_{a \geqslant -1} \mathscr{A}^a(V)$, where the space $\mathscr{A}^a(V)$ is the space of all antisymmetric elements of $M^a(V)$, define $\bar{\Delta}: \mathscr{A}(V)^2 \to \mathscr{A}(V)$ by

$$A \bar{\Delta} B = \frac{(a+b+1)!}{(a+1)!(b+1)!} \alpha(A \Delta B), \quad A \in \mathscr{A}^a(V), \quad B \in \mathscr{A}^b(V) ,$$

where α stands for the antisymmetrization projector in $M(V)$.

Proposition 2.1. *The pairs $(M(V), \Delta)$ and $(\mathscr{A}(V), \bar{\Delta})$ are graded Lie algebras.*

Remark 2.2. The "bracket" Δ was introduced by De Wilde and Lecomte. Its antisymmetric part $\bar{\Delta}$ was earlier known as Nijenhuis–Richardson bracket [26]. On the other hand, the Nijenhuis–Richardson bracket was introduced as an algebraic generalization of the Nijenhuis–Schouten bracket for antisymmetric contravariant tensors on a manifold [27].

For a graded Lie algebra $(E, \Box)$ and $A \in E$ define $\partial_A : E \to E$ by $\partial_A(B) = (-1)^b A \Box B$, $B \in E^b$.

Proposition 2.3. *Let $(E, \Box)$ be a graded Lie algebra $(M(V), \Delta)$ $[(\mathscr{A}(M), \bar{\Delta})]$. Then $A \in E^1$ defines an associative (Lie) algebra structure on V if and only if $A \Box A = 0$. In this case $\partial_A : E \to E$ is homogeneous of degree 1 and satisfies $\partial_A \circ \partial_A = 0$. Hence $\Box$ induces on the cohomology space $H(E, \Box) = \ker \partial_A / \operatorname{im} \partial_A$ a graded Lie algebra structure.*

Remark 2.4. If $A \in M(V)$ is such that $A \Delta A = 0$, we have seen that (V, A) is an associative algebra. The cohomology of ∂_A is the Hochschild cohomology of this associative algebra. If $A \in \mathscr{A}^1(V)$ such that $A \bar{\Delta} A = 0$, then (V, A) is a Lie algebra and the cohomology of ∂_A is the Chevalley cohomology of the adjoint representation of (V, A).

Denote by V_ε the space of all formal series $x_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k x_k$ $(x_k \in V)$. An element A_ε of $M^p(V_\varepsilon)$ is *formal* if it has the form

$$A_\varepsilon((x_\varepsilon^{(0)}, \ldots, x_\varepsilon^{(p)})) = \sum_{k=0}^{\infty} \varepsilon^k \left(\sum_{r+s_0+\cdots+s_p=k} A_r(x_{s_0}^{(0)}, \ldots, x_{s_p}^{(p)}) \right),$$

where $A_r \in M^p(V)$ for each $r \geqslant 0$; A_r is called *the rth component* of A_ε and we can write $A_\varepsilon = \sum_{r=0}^{\infty} \varepsilon^r A_r$. Thus the set of formal elements of $M^p(V_\varepsilon)$ or $\mathscr{A}^p(V_\varepsilon)$ identifies naturally to $M^p(V)_\varepsilon$ and $\mathscr{A}^p(V)_\varepsilon$.

Proposition 2.5. *The space $M(V)_\varepsilon$ is a graded Lie subalgebra of $(M(V_\varepsilon), \Delta)$ and the space $\mathscr{A}(V)_\varepsilon$ is a graded Lie subalgebra of $(\mathscr{A}(V_\varepsilon), \bar{\Delta})$.*

Let (V, A) be an associative or a Lie algebra. A *formal deformation* A_ε of A is an associative or a Lie algebra structure on V_ε such that A_ε is formal and $A_0 = A$. Writing $\Box$ for Δ or $\bar{\Delta}$, A_ε is an associative or a Lie algebra structure if and only if $A_\varepsilon \Box A_\varepsilon = 0$ and this in turn is equivalent to

$$\sum_{i+j=k} A_i \Box A_j = 0 \quad \text{for } k = 1, 2, \ldots .$$

A *formal deformation of order* k of (V, A) is a formal A_ε such that $A_0 = A$ and $\sum_{i+j=l} A_i \Box A_j = 0$ for all $l \leqslant k$.

Proposition 2.6. *A bilinear formal map $A_\varepsilon = \sum_{i=1}^{\infty} \varepsilon^i A_i$ is a formal deformation of order k of A_0 if and only if $2\partial_{A_0} A_i = J_i$ for all $i \leqslant k$, where*

$$J_i = \sum_{\substack{r+s=i \\ r,s>0}} A_r \Box A_s ,$$

and only if $\partial_{A_0} J_{k+1} = 0$.

In order to construct a formal deformation of A_0, a natural approach is to construct by induction A_k such that $\sum_{i=0}^{k} \varepsilon^i A_i$ is a formal deformation of order k. To pass from step k to step $k+1$, we know that J_{k+1} is a cocycle for ∂_{A_0} and that we can extend the deformation to the order $k+1$ by adding $\varepsilon^{k+1} A_{k+1}$ if and only if $J_{k+1} = 2\partial_{A_0} A_{k+1}$. Thus the obstruction to extending the deformation from order k to order $k+1$ is the cohomology class of J_{k+1}.

3. Linear differential operators in commutative algebras

In his famous paper [18] Kirillov considered Lie algebra structures on the space $C^\infty(N)$ of smooth functions on a manifold N given by differential operators. We had the impression that at least a part of his work is in fact of a purely algebraic nature and we succeeded in proving the corresponding generalization, which will be presented in the next section. It can be applied to such geometric structures as orbit or leaves spaces, for which we usually have no convenient manifold structure, but algebras of "smooth functions" can be easily defined. Thus we shall make precise in this section what differentiability means from the algebraic point of view. These ideas come from the work of Grothendieck and were used with great success in a series of papers by Vinogradov and Krasil'shchik (see, e.g., refs. [21,29]).

Our object under consideration will be an associative commutative algebra with unit $(V, m, \mathbf{1})$ over a field $\mathscr{k}$ of characteristic 0. The standard model is of course the algebra $C^\infty(N)$ of smooth functions on a manifold N, but we can also take into account algebras of polynomial, real analytic or holomorphic functions, or algebras of functions invariant with respect to a group action or functions constant on leaves of a given foliation.

We shall usually write "xy" instead of "$m(x, y)$".

The graded vector space $M(V)$ (cf. section 2) possesses the natural associative algebra structure "$\cdot$" defined by

$$A \cdot B(x_0, ..., x_{a+b+1}) = A(x_0, ..., x_a) B(x_{a+1}, ..., x_{a+b+1}) ,$$

$A \in M^a(V)$, $B \in M^b(V)$. Using the antisymmetrization projector α, we can define the wedge product in $\mathscr{A}(V)$ putting

$$A\wedge B=\frac{(a+b+2)!}{(a+1)!(b+1)!}\alpha(A\cdot B)\,,$$

which makes $\mathscr{A}(V)$ into a graded commutative (or super) algebra. Define an additional coboundary operator $d : M^a(V)\to M^{a+1}(V)$ by

$$dA(x_0,\ldots,x_{a+1})=\sum_{i=0}^{a}(-1)^i x_i A(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_{a+1})\,.$$

It is a matter of computation to show that $d^2=0$. Since $\alpha\circ d=d\circ\alpha$, d is also an operator on $\mathscr{A}(V)$ and, as can be easily seen, it is exactly the coboundary operator for the cohomology of the commutative Lie algebra V with coefficients in the regular representation induced by the multiplication m.

In $M(V)$ we can distinguish the graded subspace of differential operators as follows. In $\mathrm{End}_k(V)=M^0(V)$ we define recurrently the subspace $\mathrm{Diff}_r(V)$ of linear differential operators of order $\leqslant r$ by

(i) $\mathrm{Diff}_r(V)=\{0\}$ if $r<0$,

(ii) $\mathrm{Diff}_0(V)=\{D\in M^0(V) : D=H_x$ for some $x\in V\}$, where $H_x(y)=xy$,

(iii) $\mathrm{Diff}_{r+1}(V)=\{D\in M^0(V) : \delta(x)D\in \mathrm{Diff}_r(V)$ for all $x\in V\}$, where $\delta(x)D$ $[D,H_x]$ and the bracket $[\ ,\ ]$ is the usual commutator $[D_1,D_2]=D_1\circ D_2-D_2\circ D_1$.

Note that $\mathrm{Diff}_0(V)$ is a commutative subalgebra of $M^0(V)$ and it is exactly the space of cocycles with respect to the coboundary operator $d : M^0(V)\to M^1(V)$.

Since the operators $\{\delta(x) : x\in V\}$ commute, we can also write

$$\mathrm{Diff}_r(V)=\{D\in M^0(V) : \delta(x_0)\circ\cdots\circ\delta(x_r)D=0 \quad \text{for all } x_0,\ldots,x_r\in V\}\,.$$

It is not hard to verify that

$$\mathrm{Diff}_1(V)=\mathrm{Der}(V)\oplus\mathrm{Diff}_0(V)\,,$$

where $\mathrm{Der}(V)$ is the Lie algebra of derivations of V and the splitting is given by

$$\mathrm{Diff}_1(V)\ni D\mapsto(D-H_{D(\mathbf{1})})+H_{D(\mathbf{1})}\in\mathrm{Der}(V)\oplus\mathrm{Diff}_0(V)\,.$$

Hence

$$\mathrm{Der}(V)=\{D\in\mathrm{Diff}_1(V) : D(\mathbf{1})=0\}\,, \tag{3.1}$$

i.e., derivations are exactly differential operators vanishing on constants.

It is also clear that the space $\mathrm{Diff}(V):=\bigcup_{r\geqslant 0}\mathrm{Diff}_r(V)$ of all differential operators on V is in a natural way a filtered associative and a filtered Lie algebra, since

$$D_1\circ D_2\in\mathrm{Diff}_{n+k}(V),\quad [D_1,D_2]\in\mathrm{Diff}_{n+k-1}(V)\,,$$
$$\text{for } D_1\in\mathrm{Diff}_n(V),\quad D_2\in\mathrm{Diff}_k(V)\,. \tag{3.2}$$

We say that $D\in\mathrm{Diff}(V)$ *is of order* r [and we write $\mathrm{rank}(D)=r$], if $D\in\mathrm{Diff}_r(V)$

and $D \notin \mathrm{Diff}_{r-1}(V)$.

The introduced definition coincides with the usual one for $V = C^\infty(N)$.

Proposition 3.1. *Let V be the algebra $C^\infty(N)$ of smooth functions on a manifold N. Then $D \in \mathrm{Diff}_r(V)$ if and only if D is a local operator and in every local coordinate chart $(U, (x_1, \ldots, x_n))$ on N we have*

$$D(f)(x_1, \ldots, x_n) = \sum_{|\beta| \leqslant r} g_\beta(x_1, \ldots, x_n) D^\beta(f)(x_1, \ldots, x_n),$$

where g_β are smooth functions on U and $D^\beta = \partial^{|\beta|} / \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}$ for all multiindices $\beta = (\beta_1, \ldots, \beta_n)$.

Remark 3.2. For $V = C^\infty(N)$, one can prove that the algebra $\mathrm{Diff}(V)$ is generated by $\mathrm{Diff}_1(V)$. This is not true in general. For instance, consider the algebra $V = \mathbb{R}(x_1, x_2, \ldots)$ of polynomials in infinitely many variables. It is easy to see that

$$\mathrm{Der}(V) = \left\{ \sum_{i=1}^{\infty} a_i \frac{\partial}{\partial x_i}, a_i \in V \right\}.$$

On the other hand, $D = \sum_{i=1}^{\infty} \partial^2 / \partial x_i^2 \in \mathrm{Diff}_2(V)$ and one can easily see that D is not a polynomial in variables from $\mathrm{Diff}_1(V)$ [cf. (3.1)].

A useful tool to determine differential operators of order r is provided by the following proposition.

Proposition 3.3. *An operator $D \in M^0(V)$ is a differential operator of order $\leqslant r$ if and only if $\delta(x)^{r+1} D = 0$ for all $x \in V$.*

Proof. Take $x_0, \ldots, x_r \in V$ and $t_0, \ldots, t_r \in k$. For $x = t_0 x_0 + \cdots + t_r x_r$ we have

$$\begin{aligned} 0 &= \delta(x)^{r+1}(D) \\ &= \sum_{i_0 + \cdots + i_r = r+1} \frac{(r+1)!}{i_0! \cdots i_r!} t_0^{i_0} \cdots t_r^{i_r} \delta(x_0)^{i_0} \cdots \delta(x_r)^{i_r}(D). \end{aligned}$$

Since k is of characteristic 0, the polynomials $t_0^{i_0} \cdots t_r^{i_r}$ are linearly independent and, finding a suitable linear combination, we get that $\delta(x_0) \cdots \delta(x_r)(D) = 0$. □

The multilinear differential operators are defined recurrently in a natural way. Namely, $D \in M^p(V)$ $(p > 0)$ is a *differential operator of order* $\leqslant r$, if $i_n(x) D$ is a differential operator of order $\leqslant r$ for each $x \in V$ and $n = 0, \ldots, p$, where

$$i_n(x) D(x_0, \ldots, x_{p-1}) = D(x_0, \ldots, x_{n-1}, x, x_n, \ldots, x_{p-1}).$$

The space of all $(p+1)$-linear differential operators of order $\leqslant r$ will be denoted by $\mathrm{Diff}^p_r(V)$, the set of all $(p+1)$-linear differential operators will be denoted by $\mathrm{Diff}^p(V)$, and $\mathrm{Diff}^*(V)=\oplus_{p\geqslant -1}\mathrm{Diff}^p(V)$.

The antisymmetric differential operators will be denoted $\mathscr{A}\mathrm{Diff}^p_r(V)$, etc. For $0\leqslant n\leqslant p$ and $x\in V$ define $\delta_n(x): M^p(V)\to M^p(V)$ by

$$\delta_n(x)D(x_0,\dots,x_p)=D(x_0,\dots,x_{n-1},xx_n,x_{n+1},\dots,x_p)-xD(x_0,\dots,x_p)\,.$$

We say that $D\in M^p(V)$ is a *differential operator of order $\leqslant r$ with respect to the nth variable*, if $\delta_n(x_0)\cdots\delta_n(x_r)D=0$ for all $x_0,\dots,x_r\in V$. By proposition 3.3, this is equivalent to the fact that $\delta_n(x)^{r+1}D=0$ for all $x\in V$.

It is easy to see that $D\in M^p(V)$ is a differential operator if and only if it is a differential operator with respect to all variables and the order of D is the maximum order of D with respect to all variables. Note also that $\delta_n(x)$ and $\delta_k(y)$ commute for all $0\leqslant n,\ k\leqslant p$ and $x,\ y\in V$ and that $i_n(x)$ and $\delta_k(y)$ commute for $n\neq k$.

Let $\mathrm{Der}^p=\{D\in\mathrm{Diff}^p_1(V): i_n(\mathbf{1})D=0 \text{ for } 0\leqslant n\leqslant p\}$ be the space of all $(p+1)$-linear derivations [cf. (3.1)]. Also let $\mathrm{Der}^*(V)=\oplus_{p\geqslant -1}\mathrm{Der}^p(V)$ be the space of multiderivations and $\mathscr{A}\mathrm{Der}^p(V)$ and $\mathscr{A}\mathrm{Der}^*(V)$ the same for antisymmetric multiderivations. $\mathrm{Diff}^*(V)$ and $\mathrm{Der}^*(V)$ [$\mathscr{A}\mathrm{Diff}^*(V)$ and $\mathscr{A}\mathrm{Der}^*(V)$] are clearly associative subalgebras of $(M(V),\cdot)$ [$(\mathscr{A}(V),\wedge)$]. One can prove moreover that $D_1\bar{\Delta}D_2\in\mathscr{A}\mathrm{Diff}^{a+b}_{n+k-1}(V)$ if $D_1\in\mathscr{A}\mathrm{Diff}^a_n(V)$ and $D_2\in\mathscr{A}\mathrm{Diff}^b_k(V)$, which generalizes (3.2). Thus we get the following proposition.

Proposition 3.4.

(i) $\mathrm{Diff}^*(V)$ *is a graded Lie subalgebra of the graded Lie algebra* $(M(V),\Delta)$. *Moreover,* $\mathrm{Diff}^*(V)$ *is invariant with respect to the coboundary operators* ∂_m *and* d.

(ii) $\mathscr{A}\mathrm{Diff}^*(V)$, $\mathscr{A}\mathrm{Diff}^*_1(V)$ *and* $\mathscr{A}\mathrm{Der}^*(V)$ *are graded Lie subalgebras of the graded Lie algebra* $(\mathscr{A}(V),\bar{\Delta})$.

Proof. It is quite obvious that $i(B)A\in\mathrm{Diff}^{a+b}_{n+k}(V)$ for $A\in\mathrm{Diff}^a_n(V)$, $B\in\mathrm{Diff}^b_k(V)$. If A and B are antisymmetric, then

$$i(B)A(x_0,\dots,x_{a+b+1})=\sum_J\sigma(J)A(B(x_J),x_{J^*})\,,$$

where J runs over all subsets of $\{0,\dots,a+b+1)$ with $b+1$ elements, J^* is the complement of J, $x_J=(x_{J_1},\dots,x_{J_{b+1}})$ with $J_1<\cdots<J_{b+1}$ being elements of J and $\sigma(J)$ being the sign of the permutation transforming $(x_0,\dots,x_{a+b+1})$ into (x_J,x_{J^*}).

In order to show that $A\bar{\Delta}B$ is of order $\leqslant n+k-1$, it suffices to show that it is of order $\leqslant n+k-1$ with respect to x_0. The antisymmetric multilinear operators of order 0 are trivial, so we can assume that $n,\ k\geqslant 1$. Hence $A(B(x_J),x_{J^*})$ is of order $\leqslant n \leqslant n+k-1$ if $0 \in J^*$ and

$$\delta_0(z)^{n+k} i(B)A(x_0, \dots, x_{a+b+1}) = x_0 \sum_I \sigma(I)\delta_0(z)^k B(\mathbf{1}, x_I)\delta_0(z)^n A(\mathbf{1}, x_{I^*}) ,$$

where I runs over all subsets of $\{1, \dots, a+b+1\}$ with b elements. Hence

$$\begin{aligned}\delta_0(z)^{n+k} A \bar{\Delta} B(x_0, \dots, x_{a+b+1}) &= x_0 \sum_I [\sigma(I) + (-1)^{ab+1}\sigma(I^*)] \\ &\quad \times \delta_0(z)^k B(\mathbf{1}, x_I)\delta_0(z)^n A(\mathbf{1}, x_{I^*}) = 0 .\end{aligned}$$

Finally, $i(B)A$ vanishes on constants if A and B do. □

Remark 3.5. For V being the algebra $C^\infty(N)$ of smooth functions on a manifold N, $\mathscr{A}\mathrm{Der}^*(V)$ can be interpreted as the space of all antisymmetric contravariant tensors on N. The multiplication $\bar{\Delta}$ coincides, up to some constant factors, with the Nijenhuis–Schouten bracket.

4. Jacobi and Poisson algebras

Well-known examples of Lie algebra structures on the spaces $C^\infty(N)$ of smooth functions on manifolds are those given by the Poisson bracket on symplectic manifolds and by the Lagrange bracket on contact manifolds. Kirillov considered in ref. [18] all possible Lie algebra structures on $C^\infty(N)$ given by differential operators. He proved that such operators must be of order 1 and he described the corresponding structures, which appeared to split into Poisson and Lagrange structures given by symplectic and contact forms on leaves of generalized foliations induced on the manifolds. The algebraic part of his work is covered by results we shall prove in the framework developed in section 3. Our methods are different, but we hope that they touch the core of the problem. We call the algebraic structure *Jacobi algebras* after Lichnerowicz, who used the notion of a Jacobi structure in ref. [23] in a geometrical but, as will be seen, corresponding context.

Definition 4.1. A *Jacobi algebra* is an associative commutative algebra V with unit $\mathbf{1}$ over a field k of characteristic 0 equipped with a skew-symmetric differential operator $P \in \mathscr{A}\mathrm{Diff}^1(V)$ which defines on V a Lie algebra structure. P will also be called a *Jacobi structure* on V. A Jacobi structure will be called a *Poisson structure* on V, if P vanishes on constants, i.e., $P(\mathbf{1}, \cdot) = 0$.

Theorem 4.2. *If P is a Jacobi structure on V and V has no nontrivial nilpotent elements, then P is of order $\leqslant 1$.*

Proof. For $x\in V$ put $P_x=i_0(x)P$. The Jacobi identity is then equivalent to $[P_x, P_y]=P_{P(x,y)}$, i.e., $x\mapsto P_x$ establishes a homomorphism of the Lie algebra (V, P) into the Lie algebra $\mathrm{Diff}(V)$ of differential operators on V. Let $n=\mathrm{rank}(P)$. Hence $P_x\in \mathrm{Diff}_n(V)$ for all $x\in V$. We will show that $n=1$.

Suppose the contrary. Let

$$k=\max\{\mathrm{rank}(\,(\delta_1(z_1)\cdots\delta_1(z_n)P)(\cdot, \mathbf{1})\,) : z_1, \dots, z_n\in V\}\,.$$

Since P is of order n with respect to the second variable, $k\geqslant 0$. Similarly to proposition 3.3 one can prove that $k=\max\{\mathrm{rank}(\,(\delta_1(z)^nP)(\cdot, \mathbf{1}) : z\in V\}$ and that

$$\text{there are } u_0, z_0\in V \text{ such that } \delta_0(u_0)^k\delta_1(z_0)^nP\neq 0\,. \tag{4.1}$$

Take $x, y, z\in V$. Since P_x, P_y, $[P_x, P_y]$ are of order $\leqslant n$ and $2n-1\geqslant n+1$ $(n>1)$, we have

$$\begin{aligned}0&=\delta(z)^{2n-1}[P_x, P_y]\\ &=\binom{2n-1}{n}(\,[\delta(z)^nP_x, \delta(z)^{n-1}P_y]+[\delta(z)^{n-1}P_x, \delta(z)^nP_y])\,.\end{aligned}$$

Hence for each $w\in V$,

$$\begin{aligned}0=&\,\delta(z)^nP_x(\delta(z)^{n-1}P_y(w))-\delta(z)^{n-1}P_y(\delta(z)^nP_x(w))\\ &+\delta(z)^{n-1}P_x(\delta(z)^nP_y(w))-\delta(z)^nP_y(\delta(z)^{n-1}P_x(w))\,.\end{aligned}$$

Since $\delta(z)^nP_u=i_0(u)\delta_1(z)^nP$ is of order 0, we can rewrite the above equality in the form

$$\begin{aligned}0=&\,\delta(\delta(z)^nP_y(\mathbf{1}))\delta(z)^{n-1}P_x(w)+(\delta(z)^{n-1}P_y(w))(\delta(z)^nP_x(\mathbf{1}))\\ &-(\delta(w)\delta(z)^{n-1}P_y(\mathbf{1}))(\delta(z)^nP_x(\mathbf{1}))-w\delta(z)^{n-1}P_y(\delta(z)^nP_x(\mathbf{1}))\,.\end{aligned}$$

For fixed w, y, z the right-hand side, $\Phi(z, w, y, x)$, is a linear differential operator with respect to x. All terms, except for the last one, are clearly of order $\leqslant k$, so that

$$0=(\delta_3(u_0)\cdots\delta_3(u_k)\Phi)(z, w, y, x)=-w(\delta_3(u_0)\cdots\delta_3(u_k)\Psi)(z, y, x)\,,$$

for all $u_0, \dots, u_k\in V$, where $\Psi(z, y, x)=\delta(z)^{n-1}P_y(\delta(z)^nP_x(\mathbf{1}))$. It is easily seen that

$$\begin{aligned}(\delta_3(u)\Psi)(z, y, x)=&\,\delta_1(z)^{n-1}P(y, \delta_0(u)\delta_1(z)^nP(x, \mathbf{1}))\\ &+\delta_1(u)\delta_1(z)^{n-1}P(y, \delta_1(z)^nP(x, \mathbf{1}))\,.\end{aligned}$$

Hence, inductively,

$$\begin{aligned}0&=(\delta_3(u_0)\cdots\delta_3(u_k)\Psi)(z,y,x)\\&=\sum_{i=0}^{k}\delta_1(u_i)\delta_1(z)^{n-1}\\&\quad\times P(y,\delta_0(u_0)\cdots\delta_0(\hat{u}_i)\cdots\delta_0(u_k)\delta_1(z)^nP(x,\mathbf{1}))\,,\end{aligned}$$

where the hat stands for omission, and further

$$\begin{aligned}&\sum_{i=0}^{k}\delta_0(u_0)\cdots\delta_0(\hat{u}_i)\cdots\delta_0(u_k)\delta_1(z)^nP(\mathbf{1},\mathbf{1})\\&\quad\cdot\delta_1(u_i)\delta_1(z)^{n-1}P(\cdot,\mathbf{1})=0\end{aligned}\tag{4.2}$$

for all $u_0,\ldots,u_k$, $z\in V$, since the corresponding operators are of order 0. Putting $u_0=\cdots=u_k=z$ in (4.2), we get

$$\delta_0(z)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1})\cdot\delta_1(z)^nP(\cdot,\mathbf{1})=0\,.$$

Hence $(\delta_0(z)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1}))^2=0$ and

$$\delta_0(z)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1})=0\,,\tag{4.3}$$

since V has no nonzero nilpotent elements. Putting inductively in (4.2) $u_0=\cdots=u_j=u$, $u_{j+1}=\cdots=u_k=z$, we get finally in a similar way that

$$\delta_0(u)^{k-1}\delta_0(z)\delta_1(z)^nP(\mathbf{1},\mathbf{1})=0\,,\tag{4.4}$$

$$\delta_0(u)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1})\cdot\delta_1(u)\delta_1(z)^{n-1}P(\cdot,\mathbf{1})=0\,.\tag{4.5}$$

For $t\in\mathbb{k}$ put $z:=z+tu$ in (4.4). Similarly to the proof of proposition 3.3, the coefficients of the polynomial

$$Q(t)=\delta_0(u)^{k-1}(\delta_0(z)+t\delta_0(u))(\delta_1(z)+t\delta_1(u))^nP(\mathbf{1},\mathbf{1})$$

vanish. In particular,

$$\begin{aligned}&n\delta_0(u)^{k-1}\delta_0(z)\delta_1(u)\delta_1(z)^{n-1}P(\mathbf{1},\mathbf{1})\\&\quad+\delta_0(u)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1})=0\,.\end{aligned}$$

Multiplying both sides by $\delta_0(u)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1})$, we get

$$\begin{aligned}&n\delta_0(u)^{k-1}\delta_0(z)\delta_1(u)\delta_1(z)^{n-1}P(\mathbf{1},\mathbf{1})\cdot\delta_0(u)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1})\\&\quad+(\delta_0(u)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1}))^2=0\,.\end{aligned}$$

The first term vanishes by (4.5), so $\delta_0(u)^k\delta_1(z)^nP(\mathbf{1},\mathbf{1})=0$ for all u, $z\in V$, which contradicts (4.1), since

$$\delta_0(u_0)^k\delta_1(z_0)^nP(x,y)=xy\delta_0(u_0)^k\delta_1(z_0)^nP(\mathbf{1},\mathbf{1})\,.\qquad\square$$

Remark 4.3. The assumption concerning nilpotent elements is essential. For instance, let V be freely generated by $\{x, y\}$ with the condition $x^2=0$. Then

$$P(u, v)=x\left(\frac{\partial^n}{\partial y^n}(u)v-\frac{\partial^n}{\partial y^n}(v)u\right)$$

defines a Lie algebra structure on V and P is clearly of order n.

Let now P be a Jacobi structure on V. We know now that $P\in\mathscr{A}\mathrm{Diff}_1^1(V)$. By (3.1), $D_x:=P_x-H_{P_x(1)}\in\mathrm{Der}(V)$ for each $x\in V$. Define $\Omega\in\mathscr{A}^1(V)$ by

$$\Omega(x, y)=P(x, y)-xP_1(y)+yP_1(x)\,.$$

Ω is a bilinear derivation, since $\Omega(x,\cdot)=D_x-xD_1$, and we can write $P=\Omega+dD$, where $D=D_1$ and d is the coboundary operator defined in section 3. Since P defines a Lie algebra structure, $P\bar{\Delta}P=0$, or equivalently

$$\Omega\bar{\Delta}\Omega+2\Omega\bar{\Delta}dD+dD\bar{\Delta}dD=0\,. \tag{4.6}$$

One can easily check that $dD\bar{\Delta}dD=0$ for $D\in\mathrm{Der}(V)$ and that $\Omega\bar{\Delta}dD=d(\Omega\bar{\Delta}D)-D\wedge\Omega$ for all $\Omega\in\mathrm{Der}^1(V)$, $D\in\mathrm{Der}(V)$, so (4.6) is equivalent to

$$\Omega\bar{\Delta}\Omega+2d(\Omega\bar{\Delta}D)-2D\wedge\Omega=0\,. \tag{4.7}$$

It is easy to see that for $A\in\mathrm{Der}^*(V)$ we have $i_0(1)A=0$ and $i_0(1)dA=A$, so applying $i_0(1)$ to (4.7) we get $\Omega\bar{\Delta}D=0$. Thus (4.7) is equivalent to the system of equations

(i) $D\bar{\Delta}\Omega=0$,

(ii) $\Omega\bar{\Delta}\Omega=2D\wedge\Omega$.

Since in the case $V=C^\infty(N)$ and for $D\in\mathrm{Der}(V)$ the map $\mathrm{Der}^*(V)\ni A\mapsto D\bar{\Delta}A\in\mathrm{Der}^*(V)$ corresponds to the so-called Lie derivative along the vector field D (which is usually written as $\mathscr{L}_D$), we get algebraically the same formulas as Kirillov in ref. [18] and Lichnerowicz in ref. [23].

We summarize the results in the following theorem.

Theorem 4.4. *Suppose that V is an associative commutative algebra with unit over a field of characteristic 0 containing no nonzero nilpotent elements. Then the following are equivalent:*

(i) $P\in\mathscr{A}\mathrm{Diff}^1(v)$ is a Jacobi structure on V.

(ii) $P=\Omega+dD$, where $\Omega\in\mathrm{Der}^1(V)$, $D\in\mathrm{Der}(V)$ satisfy the conditions (a) $D\bar{\Delta}\Omega=0$, (b) $\Omega\bar{\Delta}\Omega=2D\wedge\Omega$. Moreover, $D=0$ if and only if P is a Poisson structure.

From now on we will deal with algebras V containing no nonzero nilpotent elements and hence, by theorem 4.4, with Poisson structures given by bilinear derivations. In this case our algebraic concept agrees with the usual one due to Krasil'shchik and Vinogradov [21] (they call this object the canonical algebra) and Guillemin and Sternberg [16]. Graded versions of this definition are due to

Braconnier [6] and Krasil'shchik [19,20].

It is easy to see $(\mathscr{A}\mathrm{Der}^*(V), \wedge, \bar{\Delta})$ is a graded Poisson algebra. We also have a Poisson algebra structure on the space of symmetric multiderivations (cf. ref. [4]).

Let now P be a Poisson structure on an algebra V. Since P is a bilinear derivation, it is easy to see that $\partial_m P=0$, i.e., P is a Hochschild cocycle. Thus $m+\varepsilon P$ is a formal deformation of the multiplication m in V of order one. We will look for special formal deformations of m of higher orders.

Definition 4.5. A formal deformation $A_\varepsilon = m+\varepsilon P+\sum_{k=2}^{\infty} \varepsilon^k A_k$ is called a *star-product for P* if, for each $k>1$,

(i) $A_k \in \mathrm{Diff}^1(V)$,

(ii) $A_k(u,v) = (-1)^k A_k(v,u)$,

(iii) A_k is vanishing on constants, i.e., $i_0(\mathbf{1})A_k=0$.

The assumption that A_k vanish on constants ensures that $\mathbf{1}$ remains the unit in the associative algebra $(V_\varepsilon, A_\varepsilon)$.

A significant example of a Poisson structure generated by no symplectic form is the canonical Lie–Poisson structure on the dual $\mathscr{L}^*$ of a Lie algebra $\mathscr{L}$ discovered by Lie. The orbits of the coadjoint action of the corresponding Lie group are exactly the symplectic leaves of this structure, which was rediscovered by Berezin and, finally, by Kirillov and Souriau.

Regarding elements from $\mathscr{L}$ as functionals on $\mathscr{L}^*$, we can write $P(x,y)=[x,y]$, where $[\ ,\ ]$ is the Lie bracket in $\mathscr{L}$. Since any Poisson structure on $C^\infty(\mathscr{L}^*)$ is a bilinear derivation, it is completely described by the action on functionals. In local coordinates $x_1, \dots, x_n \in \mathscr{L}$ on $\mathscr{L}^*$ we can write

$$P=\sum_{i,j} [x_i, x_j]\, \partial_i \wedge \partial_j ,$$

where $\partial_i = \partial/\partial_{x_i}$.

Gutt [17] observed that a star-product for P is actually given by the multiplication in the universal enveloping algebra $\mathscr{U}=\mathscr{U}(\mathscr{L})$ or, in other words, that the universal enveloping algebra is in fact a star-product (quantization) of P.

Drinfel'd used in ref. [12] a direct formula for this star-product without mentioning it explicitly. Since the formula seems not to be widespread, we would like to present a very short proof of it.

Let $x_1, \dots, x_m$ be a basis of a Lie algebra $\mathscr{L}$. The symmetric algebra $S=S(\mathscr{L})$ can then be naturally identified with the algebra of polynomials on $\mathscr{L}^*$ and thus regarded as embedded in $C^\infty(\mathscr{L}^*)$. On the other hand, S and $\mathscr{U}$ are naturally isomorphic as vector spaces via the symmetrization mapping. Let us add a free formal parameter ε putting the Lie bracket in $\mathscr{L}_\varepsilon$ to be $\varepsilon[\ ,\ ]$. We have the multiplication $*$ on $\mathscr{U}_\varepsilon$, which can be understood as an associative structure on S_ε (cf. Dixmier [11]).

Theorem 4.6 (Gutt, Drinfel'd). *$(S_\varepsilon, *)$ is a star-product for the canonical Poisson structure on the dual $\mathscr{L}^*$. The multiplication $*$ can be written explicitly in the form*

$$f*g=fg+\sum_{n=1}^{\infty}\frac{1}{n!}$$
$$\times \sum_{|\alpha_i|+|\beta_i|>1} c_{\alpha_1\beta_1}\cdots c_{\alpha_n\beta_n}\varepsilon^{\Sigma(|\alpha_i|+|\beta_i|)-n}\partial^{\Sigma\alpha_i}(f)\partial^{\Sigma\beta_i}(g)\,,$$

where, for multiindices $\alpha_i=\alpha=(\alpha^1,\ldots,\alpha^m)$, $\beta_i=\beta=(\beta^1,\ldots,\beta^m)$, the functional $c_{\alpha\beta}$ as an element from $\mathscr{L}$ is the coefficient in the Campbell–Baker–Hausdorff series,

$$\mathrm{CH}(\Sigma\, t_k x_k, \Sigma\, s_j x_j)=\sum t_k x_k+\sum s_j x_j+\tfrac{1}{2}\varepsilon\sum t_k s_j[x_k, x_j]+\cdots\,,$$

of the term

$$t^\alpha s^\beta=t_1^{\alpha^1}\cdots t_m^{\alpha^m}s_1^{\beta^1}\cdots s_m^{\beta^m}$$

and ∂^α denotes $\partial_1^{\alpha^1}\cdots\partial_m^{\alpha^m}$.

Proof. Let x and y be elements of a (e.g. free) Lie algebra $\mathscr{L}_\varepsilon$ with the bracket $\varepsilon[\ ,\ ]$. In the universal enveloping algebra $(\mathscr{U}_\varepsilon, *)$ we can write

$$\mathrm{e}^{tx}*\mathrm{e}^{sy}=\mathrm{e}^{\mathrm{CH}(tx,sy)}\,,$$

where $\mathrm{CH}(tx, sy)=\Sigma\, t^\alpha s^\beta c_{\alpha\beta}$ is the Campbell–Baker–Hausdorff series. We have

$$\mathrm{e}^{tx}*\mathrm{e}^{sy}=\sum\frac{1}{l!h!}t^l s^h x^{*l}*y^{*h}\,,$$

and since $x^{*l}=x^l$, $y^{*h}=y^h$ are symmetric we can get the symmetric form of x^l*y^h (i.e. on the level of the symmetric algebra S_ε) just by looking at the coefficient $t^l s^h$ on the right-hand side (which is clearly symmetric). This implies

$$x^l*y^h=l!h!\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{\Sigma\alpha_k=l\\ \Sigma\beta_k=h}} c_{\alpha_1\beta_1}\cdots c_{\alpha_n\beta_n}\varepsilon^{l+h-n}\,.$$

Since $c_{10}=x$ and $c_{01}=y$, it is easy to check that the right-hand term equals

$$x^l y^h+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{|\alpha_k|>0\\ |\beta_k|>0}} c_{\alpha_1\beta_1}\cdots c_{\alpha_n\beta_n}\varepsilon^{\Sigma(\alpha_k+\beta_k)-n}\partial_x^{\Sigma\alpha_k}(x^l)\,\partial_y^{\Sigma\beta_k}(y^h)\,.$$

Putting $\Sigma\, t_k x_k$ instead of tx and $\Sigma\, s_j x_j$ instead of sy and passing to multiindices we get the desired formula. □

Remark 4.7. In a more transparent form

$$f*g=\exp[(\mathrm{CH}(a,b)-a-b)/\varepsilon](f\otimes g),$$

where $a=\sum x_k\otimes\partial_k\otimes\mathrm{id}$, $b=\sum x_j\otimes\mathrm{id}\otimes\partial_j$, Lie brackets and the exponential are from the associative algebra $\mathscr{U}_\varepsilon\otimes\mathrm{Diff}\otimes\mathrm{Diff}$, Diff being the associative algebra of differential operators on $\mathscr{L}^*$ with constant coefficients, and

$$u\otimes A\otimes B\in\mathscr{U}\otimes\mathrm{Diff}\otimes\mathrm{Diff}\approx S\otimes\mathrm{Diff}\otimes\mathrm{Diff}\subset C^\infty(\mathscr{L}^*)\otimes\mathrm{Diff}\otimes\mathrm{Diff}$$

acts on $f\otimes g$ by $u\otimes A\otimes B(f\otimes g)=uA(f)B(g)$.

The first terms in the formula are as follows:

$$\begin{aligned}
f*g=&fg+\frac{\varepsilon}{2}\sum[x_i,x_j]\,\partial_i(f)\partial_j(g)\\
&+\frac{\varepsilon^2}{8}\sum[x_i,x_j][x_k,x_l]\,\partial_i\partial_k(f)\partial_j\partial_l(g)\\
&+\frac{\varepsilon^2}{12}\sum[x_k,[x_j,x_i]]\,(\partial_k\partial_j(f)\partial_i(g)+\partial_i(f)\partial_k\partial_j(g))\\
&+o(\varepsilon^2)\\
=&fg+\frac{\varepsilon}{2}\sum c_{ij}^n x_n\,\partial_i(f)\partial_j(g)\\
&+\frac{\varepsilon^2}{8}\sum c_{ij}^n c_{kl}^h x_n x_h\,\partial_i\partial_k(f)\partial_j\partial_l(g)\\
&+\frac{\varepsilon^2}{12}\sum c_{kl}^n c_{ji}^l x_n(\partial_k\partial_j(f)\partial_i(g)+\partial_i(f)\partial_k\partial_j(g))\\
&+o(\varepsilon^2),
\end{aligned}$$

where c_{ij}^n are the structure constants of $\mathscr{L}$.

The next part of this note will be devoted to problems concerning the existence and description of star-products.

It seems that we cannot retain all generality to obtain reasonable results. Therefore we shall suppose that the Poisson structure P is of the form $P=\sum_{i,j}a_{ij}D_i\wedge D_j$, where $a_{ij}\in\mathscr{A}$ and $D_i\in\mathrm{Der}(V)$. It is not a very restrictive assumption, since every $P\in\mathscr{A}\mathrm{Der}^1(V)$ is of this form for most algebras V which can appear in applications, e.g., for algebras of smooth functions on manifolds. Since we shall look for an explicit formula for star-products which would be good for every algebra V, we can make use only of the properties of derivations D_i, i.e., only of formal properties of derivations and the structure of the Lie algebra $\mathscr{L}$ generated by $\{D_i\}$. Therefore we must work on a "formal level", i.e., we can only use the tensor algebra of the universal enveloping algebra of $\mathscr{L}$. As a result we get, in some cases, a formula for star-products on the level of this tensor algebra which can be easily translated into the language of differential operators.

The next section is therefore devoted to a description of the necessary properties of the tensor algebra $\mathscr{U}^{\otimes}=\oplus_{n=1}^{\infty}\mathscr{U}^{\otimes n}$ of a universal enveloping algebra over a Lie algebra. We introduce a graded Lie algebra structure on $\mathscr{U}^{\otimes}$ compatible with the graded Lie algebra structure on $\mathrm{Diff}^*(V)$ and the notion of a Poisson element. We show that Poisson elements are exact solutions of the classical Yang–Baxter equation. We introduce and compute analogs of Hochschild cohomology as well.

5. The calculus on tensor algebras of universal enveloping algebras

The set $\mathrm{Der}(V)$ of derivations of an associative commutative algebra has a natural structure of a Lie algebra. On the other hand, any Lie algebra $\mathscr{L}$ (over a field of characteristic 0) has always a faithful representation in $\mathrm{Der}(V)$ for V being the dual $\mathscr{U}^*$ of the universal enveloping algebra $\mathscr{U}=\mathscr{U}(\mathscr{L})$ (see, e.g., ref. [11]). The associative structure on $\mathscr{U}^*$ is defined by

$$\langle fg,u\rangle=\langle f\otimes g,c(u)\rangle \quad \text{for all } f,g\in\mathscr{U}^*,\ u\in\mathscr{U}\,,$$

where $c:\mathscr{U}\to\mathscr{U}\otimes\mathscr{U}$ is the coproduct in $\mathscr{U}$, and the representation $R:\mathscr{L}\to\mathrm{Der}(\mathscr{U}^*)$ is given by

$$\langle R(x)(f),u\rangle=\langle f,xu\rangle\ .$$

R can be clearly extended to an associative algebra representation $R:\mathscr{U}\to\mathrm{Diff}(\mathscr{U}^*)$ and since it is obvious how to define the extension $R:\mathscr{U}^{\otimes}\to\mathrm{Diff}^*(\mathscr{U}^*)$, which is a mapping of graded vector spaces for $\mathscr{U}^{\otimes}$ being the tensor algebra $\mathscr{U}^{\otimes}=\oplus_{n=1}^{\infty}\mathscr{U}^{\otimes n}$, we will show how to define a graded Lie algebra structure Δ on $\mathscr{U}^{\otimes}$ for which R is a homomorphism of graded Lie algebras. Since it makes sense for an arbitrary bialgebra, we start in all generality.

Let $\mathscr{U}$ be a bialgebra in the sense of Bourbaki [5] over a field ℓ of characteristic 0, i.e, an associative algebra with unit $\mathbb{1}$ equipped with a coproduct $c:\mathscr{U}\to\mathscr{U}\otimes_{\ell}\mathscr{U}$ which is a homomorphism of associative algebras ($\mathscr{U}\otimes\mathscr{U}$ has the obvious associative algebra structure) and which is coassociative, i.e.,

$$(\mathrm{id}\otimes c)\circ c=(c\otimes\mathrm{id})\circ c\,.$$

A standard example is the universal enveloping algebra $\mathscr{U}(\mathscr{L})$ of a Lie algebra $\mathscr{L}$ for which the comultiplication has the form

$$c(D)=D\otimes\mathbb{1}+\mathbb{1}\otimes D \quad \text{for } D\in\mathscr{L}\subseteq\mathscr{U}\,.$$

All this has a slight generalization. Namely, we have

$$c^k : \mathscr{U} \to \mathscr{U}^{k+1}, \quad \mathscr{U}^{k+1} = \mathscr{U}^{\otimes(k+1)} = \underbrace{\mathscr{U} \otimes \cdots \otimes \mathscr{U}}_{k+1 \text{ times}},$$

which is a homomorphism of associative algebras defined inductively by

$$c^{k+1} = (\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c) \circ c^k, \qquad c^1 = c.$$

The ordering of id and c in the definition of c^k does not matter because of the coassociativity of c.

For the universal enveloping bialgebra $\mathscr{U}(\mathscr{L})$ we have

$$c^k(D) = D \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes D \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes D$$

for $D \in \mathscr{L}$. In other words, c^k is the homomorphism induced by the diagonal homomorphism of Lie algebras $\mathscr{L} \to \mathscr{L} \times \cdots \times \mathscr{L}$. Elements D in $\mathscr{U}$ for which $c^k(D) = D \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes D \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes D$ are called *primitive* and it is well known that the set of primitive elements coincides with $\mathscr{L}$.

For $0 \leqslant i \leqslant p$ and $0 \leqslant j \leqslant p+1$, define $I_j^k, c_i^k : \mathscr{U}^{p+1} \to \mathscr{U}^{p+k+1}$ by

$$c_i^k(x_0 \otimes \cdots \otimes x_p) = x_0 \otimes \cdots \otimes x_{i-1} \otimes c^k(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_p,$$

$$I_j^k(x_0 \otimes \cdots \otimes x_p) = x_0 \otimes \cdots \otimes x_{j-1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes x_j \otimes \cdots \otimes x_p,$$

It is obvious that c_i^k and I_j^k are homomorphisms of associative algebras and that $c_i^b \circ c^a = c^{a+b}$ for all $0 \leqslant i \leqslant a$ because of the coassociativity of c.

For $A \in \mathscr{U}^{a+1}$, $B \in \mathscr{U}^{b+1}$ put now

$$i(B)A = \sum_{k=0}^{a} (-1)^{kb} c_k^b(A) \cdot I_0^k(I_{b+1}^{a-k}(B)),$$

where "$\cdot$" stands for multiplication in $\mathscr{U}^{a+b+1}$, and

$$A \triangle B = i(B)A + (-1)^{ab+1} i(A)B.$$

For antisymmetric tensors $A \in \Lambda^{a+1}\mathscr{U}$ and $B \in \Lambda^{b+1}\mathscr{U}$ define the antisymmetric products by

$$A \bar{\triangle} B = \frac{(a+b+1)!}{(a+1)!(b+1)!} \alpha(A \triangle B),$$

where α is the antisymmetrization projector.

Theorem 5.1. *Let $\mathscr{U}$ be a bialgebra. Then the pairs $(\mathscr{U}^{\otimes}, \triangle)$ and $(\Lambda^*\mathscr{U}, \bar{\triangle})$ are graded Lie algebras with elements from $\mathscr{U}^{k+1}$ and $\Lambda^{k+1}\mathscr{U}$ being of rank k.*

Proof. Observe first that, for $B \in \mathscr{U}^{b+1}$, $i(B) : \mathscr{U}^{\otimes} \to \mathscr{U}^{\otimes}$ is a graded derivation of the tensor algebra $(\mathscr{U}^{\otimes}, \otimes)$ of rank b, i.e., $i(B) : \mathscr{U}^a \to \mathscr{U}^{a+b}$ and for $A \in \mathscr{U}^{a+1}$, $C \in \mathscr{U}^{c+1}$ we have

$$i(B)A\otimes C=i(B)A\otimes C+(-1)^{(a+1)b}A\otimes i(B)C\,.$$

Since the graded derivations form a graded Lie algebra and since $i(B)=0$ if and only if $B=0$ $[i(B)\mathbb{1}=B]$, it suffices to show that for $A\in\mathcal{U}^{a+1}$, $B\in\mathcal{U}^{b+1}$ the graded commutator

$$i(B)\circ i(A)+(-1)^{ab+1}i(A)\circ i(B)$$

equals $i(A\triangle B)$. Moreover, it suffices to check it only on generators of the tensor algebra, i.e. on $\mathcal{U}$.

Let $X\in\mathcal{U}$. We have

$$\begin{aligned}i(B)\circ i(A)X&=i(B)(c^a(X)\cdot A)\\&=\sum_{k=0}^{b}(-1)^{kb}c_k^b(c^a(X))\cdot c_k^b(A)\cdot I_0^k(I_{b+1}^{a-k}(B))\,.\end{aligned}$$

Since $c_k^b(c^a(X))=c^{a+b}(X)$ for all $0\leqslant k\leqslant a$, we have for $X\in\mathcal{U}$

$$i(B)\circ i(A)X=c^{a+b}(X)\cdot i(B)A\,,$$

$$\begin{aligned}&i(B)\circ i(A)X+(-1)^{ab+1}i(A)\circ i(B)X\\&=c^{a+b}(X)\cdot(A\triangle B)=i(A\triangle B)X\,.\end{aligned}$$

□

Since $(\mathbb{1}\otimes\mathbb{1})\triangle(\mathbb{1}\otimes\mathbb{1})=0$, similarly to section 2, we get a coboundary operator $\partial_{\mathbb{1}\otimes\mathbb{1}}:\mathcal{U}^{a+1}\to\mathcal{U}^{a+2}$ setting $\partial_{\mathbb{1}\otimes\mathbb{1}}A=(-1)^a(\mathbb{1}\otimes\mathbb{1})\triangle A$, or explicitly

$$\begin{aligned}&\partial_{\mathbb{1}\otimes\mathbb{1}}(x_0\otimes\cdots\otimes x_a)\\&=(-1)^a\sum_{k=0}^{a}(-1)^k c_k(A)-x_0\otimes\cdots\otimes x_a\otimes\mathbb{1}+(-1)^{a+1}\mathbb{1}\otimes x_0\otimes\cdots\otimes x_a\\&=(-1)^a\sum_{k=0}^{a}(-1)^k x_0\otimes\cdots\otimes c_{\mathbb{1}}(x_k)\otimes\cdots\otimes x_a\,,\end{aligned}$$

where $c_{\mathbb{1}}(x)=c(x)-(\mathbb{1}\otimes x+x\otimes\mathbb{1})$. This shows that the cohomology of $\partial_{\mathbb{1}\otimes\mathbb{1}}$ depends only on the coalgebra structure of the bialgebra $(\mathcal{U},c)$. One can check that $c_{\mathbb{1}}$ is a new coassociative coalgebra structure.

Theorem 5.2 *Let $(\mathcal{U},c)$ be a coassociative coalgebra.*

(i) If $e\in\mathcal{U}$ satisfies $c(e)=e\otimes e$, then $c_e:\mathcal{U}\to\mathcal{U}\otimes\mathcal{U}$ defined by $c_e=c-(e\otimes\mathrm{id}+\mathrm{id}\otimes e)$ is another coassociative coalgebra structure on $\mathcal{U}$.

(ii) The map $\partial_c:\mathcal{U}^*\to\mathcal{U}^*$, where $\mathcal{U}^*$ is the tensor algebra over $\mathcal{U}$ and ∂_c is defined by

$$\partial_c(x_0\otimes\cdots\otimes x_a)=(-1)^a\sum_{k=0}^{a}(-1)^k x_0\otimes\cdots\otimes c(x_k)\otimes\cdots\otimes x_a\,,$$

is a coboundary operator, i.e., $\partial_c^2=0$.

Proof. By straightforward computation. □

Now we can introduce the cohomology of a given coassociative coalgebra $(\mathscr{U}, c)$ putting $H(\mathscr{U}, c)=\ker \partial_c / \operatorname{im} \partial_c$. The following result describing the cohomology of a coalgebra $\mathscr{U}(\mathscr{L})$ can be regarded as a generalization of the classical result of Vey [28] about the differentiable Hochschild cohomology of the algebras of smooth functions on manifolds.

Theorem 5.3. *Let* $(\mathscr{U}, c)$ *be the universal enveloping bialgebra of a Lie algebra* $\mathscr{L}$ *over a field of characteristic zero. Then the cohomology space* $H^p(\mathscr{U}, c_1)$ *is isomorphic to the space* $\Lambda^p\mathscr{L}$ *of antisymmetric tensors over* $\mathscr{L}$. *More precisely, if* $A\in\mathscr{U}^p$ *is a cocycle, then its antisymmetric part* $\alpha(A)$ *belongs to* $\Lambda^p\mathscr{L}$ *and* $A-\alpha(A)$ *is a coboundary.*

Proof. The coalgebra structure of $(\mathscr{U}, c)$ is known to be isomorphic to the coalgebra structure $(S(\mathscr{L}), c)$ of the symmetric algebra $S(\mathscr{L})$ over the vector space $\mathscr{L}$ (Bourbaki [5]), which is a generalized version of the Poincaré–Birkhoff–Witt theorem. In other words, we can consider the Lie algebra $\mathscr{L}$ to be commutative. Since the isomorphism preserves unity, also the coalgebras $(\mathscr{U}, c_1)$ and $(S(\mathscr{L}), c_1)$ are isomorphic, so $H^p(\mathscr{U}, c_1)\approx H^p(S(\mathscr{L}), c_1)$.

Looking carefully at the proof of the theorem describing the differentiable Hochschild cohomology of the algebra $C^\infty(N)$ in ref. [7], one can see that the authors make all the computations on the level of some algebra of polynomials which is nothing but the symmetric algebra over some finite-dimensional vector space. More precisely, the $C^\infty(N)$ module of multilinear differential operators on $C^\infty(N)$ is locally freely generated by the subalgebra S of polynomials with respect to $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and we have (locally) an isomorphism $H^p_{\mathrm{diff}}(C^\infty(N), \delta)\approx H^p(S, c_1)$, where δ is the Hochschild coboundary operator for $C^\infty(N)$ and c is the natural coalgebra structure on S. The fact that we deal with polynomials in a finite set of variables plays no role in the proof, so we can just repeat the proof of Cahen, De Wilde and Gutt [7] to get $H^p(S(\mathscr{L}), c_1)\approx\Lambda^p\mathscr{L}$. □

From now on $\mathscr{U}$ denotes the universal enveloping bialgebra of a Lie algebra $\mathscr{L}$ of finite dimension over a field k of characteristic 0. Let V be an associative commutative algebra over k and let $J: \mathscr{L}\to \mathrm{Der}(V)$ be a Lie algebra homomorphism. J extends to a homomorphism $J: \mathscr{U}\to \mathrm{Diff}(V)$ of associative algebras and further to a homomorphism $J: \mathscr{U}^\otimes\to \mathrm{Diff}^*(V)$ of the tensor algebra $(\mathscr{U}^\otimes, \otimes)$ into the associative algebra $(\mathrm{Diff}^*(V), \cdot)$ (cf. section 3) by

$$J(X_0\otimes\cdots\otimes X_p)=J(X_0)\cdot\cdots\cdot J(X_p)\ .$$

We also have an induced homomorphism of graded commutative algebras $J: \Lambda^*\mathscr{U}\to\mathscr{A}\mathrm{Diff}^*(V)$. Obviously, $J(\mathbb{1}\otimes\mathbb{1})=m$ is the multiplication in V.

Theorem 5.4. *$J:\mathscr{U}^{\otimes}\to\mathrm{Diff}^*(V)$ is a homomorphism of graded Lie algebras $(\mathscr{U}^{\otimes},\Delta)$ and $(\mathrm{Diff}^*(V),\Delta)$. The same is valid for $(\Lambda^*\mathscr{U},\bar{\Delta})$ and $(\mathscr{A}\mathrm{Diff}^*(V),\bar{\Delta})$.*

Proof. It suffices to show that $J(i(B)A)=i(J(B))J(A)$, which can be reduced to the proof of the equality

$$J(c^b(A)\cdot B)=i(J(B))J(A)$$

for $A\in\mathscr{U}$ and $B\in\mathscr{U}^{b+1}$. Since $c^b:\mathscr{U}\to\mathscr{U}^{b+1}$ and $J:\mathscr{U}^{\otimes}\to\mathrm{Diff}^*(V)$ are homomorphisms of associative algebras, it suffices to take A from $\mathscr{L}$. Then

$$\begin{aligned}J(c^b(A)\cdot X_0\otimes\cdots\otimes X_b)&=J(AX_0\otimes X_1\otimes\cdots\otimes X_b+\cdots+X_0\otimes\cdots\otimes X_{b-1}\otimes AX_b)\\&=J(A)J(X_0)\cdot J(X_1)\cdot\cdots\cdot J(X_b)+\cdots+J(X_0)\cdot\cdots\cdot J(X_{b-1})\cdot J(A)J(X_b)\ .\end{aligned}$$

By definition

$$\begin{aligned}&(J(A)J(X_0)\cdot J(X_1)\cdot\cdots\cdot J(X_b)+\cdots\\&\quad+J(X_0)\cdot\cdots\cdot J(X_{b-1})\cdot J(A)J(X_b))(u_0,\dots,u_b)\\&=J(A)J(X_0)(u_0)\cdots J(X_b)(u_b)+\cdots\\&\quad+J(X_0)(u_0)\cdots J(A)J(X_b)(u_b)\ ,\end{aligned}$$

and since $J(A)$ is a derivation, the last line equals

$$\begin{aligned}&J(A)(J(X_0)(u_0)\cdots J(X_b)(u_b))\\&\quad=i(J(X_0)\cdot\cdots\cdot J(X_b))J(A)(u_0,\dots,u_b)\ .\end{aligned}$$

□

In view of section 4 it is now natural to introduce the following definition.

Definition 5.5. We call an element $\mathbb{P}$ from $\Lambda^2\mathscr{U}_+$, where $\mathscr{U}_+$ is the ideal in $\mathscr{U}$ generated by $\mathscr{L}$, a *Poisson element*, if $\mathbb{P}\bar{\Delta}\mathbb{P}=0$.

Theorem 5.6. *$\mathbb{P}$ is a Poisson element if and only if $\mathbb{P}\in\Lambda^2\mathscr{L}$ and $\mathbb{P}$ satisfies the classical Yang–Baxter equation*

$$[\mathbb{P}_0,\mathbb{P}_1]+[\mathbb{P}_0,\mathbb{P}_2]+[\mathbb{P}_1,\mathbb{P}_2]=0\ ,$$

where we write for convenience $\mathbb{P}_k$ *instead of* $I_k(\mathbb{P})$ *and the brackets denote the usual commutators in the associative algebra* $\mathscr{U}^{\otimes 3}$.

Proof. Instead of proving directly that $\mathbb{P}\in\Lambda^2\mathscr{L}$, we will make use of theorem 4.2. Consider the coregular representation $R:\mathscr{L}\to\mathrm{Der}(\mathscr{U}^*)$ and the induced homomorphism $R:(\Lambda^*\mathscr{U},\bar{\Delta})\to(\mathscr{D}\mathrm{iff}^*(\mathscr{U}^*),\bar{\Delta})$. Since $R(\mathbb{P})\bar{\Delta}R(\mathbb{P})=0$ and $R(\mathbb{P})$ vanishes on constants, $R(\mathbb{P})$ is a bilinear derivative by theorem 4.2 ($\mathscr{U}^*$ contains no nilpotent elements), i.e.,

$$R(\mathbb{P})(xy,z)=xR(\mathbb{P})(y,z)+yR(\mathbb{P})(x,z)\ .$$

This in turn is equivalent to $c_0(\mathbb{P})=\mathbb{P}_0+\mathbb{P}_1$ (recall that $\mathbb{P}_0=\mathbb{1}\otimes\mathbb{P}$, $\mathbb{P}_2=\mathbb{P}\otimes\mathbb{1}$, and $\mathbb{P}_1$ has $\mathbb{1}$ inside), which means that components of $\mathbb{P}$ consist of primitive elements. Observe now that, since $c_0(\mathbb{P})=\mathbb{P}_0+\mathbb{P}_1$ and $c_1(\mathbb{P})=\mathbb{P}_1+\mathbb{P}_2$, we have

$$\mathbb{P}\Delta\mathbb{P}=2(c_0(\mathbb{P})\mathbb{P}_2-c_1(\mathbb{P})\mathbb{P}_0)=2([\mathbb{P}_0,\mathbb{P}_2]+\mathbb{P}_1\mathbb{P}_2-\mathbb{P}_1\mathbb{P}_0)\ ,$$

where the products and commutators are taken in the associative algebra $\mathscr{U}^{\otimes 3}$. After antisymmetrization we get

$$0=\mathbb{P}\bar{\Delta}\mathbb{P}=([\mathbb{P}_0,\mathbb{P}_1]+[\mathbb{P}_0,\mathbb{P}_2]+[\mathbb{P}_1,\mathbb{P}_2])\ . \qquad \square$$

6. Star-products for Poisson elements

In this section we introduce and describe a star-product for a Poisson element which is a quantization of the classical Yang–Baxter equation. The form of this star-product follows from a result of Drinfel'd [12] and can be viewed as a generalization of the Moyal product. This formal star-product derives a star-product for some, we would say computable, Poisson structures. It only depends on algebraic properties of the Poisson structure. Since no geometry is involved, the method covers geometrically degenerate cases as well.

Note, finally, that the Moyal product is usually described as an exponential of the corresponding Poisson structure. In fact, this has a precise and unambiguous meaning on the tensor product level only.

Let us consider "formal deformations" of $\mathbb{1}\otimes\mathbb{1}\in\mathscr{U}^2$. Similarly to section 2, define a *formal deformation of* $\mathbb{1}\otimes\mathbb{1}$ to be a formal series $\mathbb{A}_\varepsilon=\mathbb{1}\otimes\mathbb{1}+\sum_{k=1}^{\infty}\varepsilon^k A_k$, where $A_k\in\mathscr{U}^2$ and $\mathbb{A}_\varepsilon\Delta\mathbb{A}_\varepsilon=0$. We define *formal deformations of order k* in an obvious way and we clearly have an analog of (2.6) with $\partial_{\mathbb{1}\otimes\mathbb{1}}$ instead of ∂_{A_0}.

Definition 6.1. Let $\mathbb{P}$ be a Poisson element of the tensor algebra of the universal enveloping algebra $\mathscr{U}$ over some Lie algebra $\mathscr{L}$. A formal deformation $\mathbb{A}_\varepsilon=\mathbb{1}\otimes\mathbb{1}+\varepsilon\mathbb{P}+\sum_{k=2}^{\infty}\varepsilon^k A_k$ is called a *star-product for* $\mathbb{P}$ if, for every $k>1$,

(i) A_k is antisymmetric if k is odd and symmetric if k is even,
(ii) A_k is "vanishing on constants", i.e., $A_k \in \mathscr{U}^2_+$.

Note that, if $\mathbb{A}_\varepsilon = \mathbb{1}\otimes\mathbb{1} + \varepsilon\mathbb{P} + \sum_{k=2}^{\infty} \varepsilon^k A_k$ is a star-product, then $\mathbb{P}_\varepsilon = \mathbb{P} + \sum_{k=1}^{\infty} \varepsilon^k A_{2k+1}$ is a formal deformation of $\mathbb{P}$, i.e., $\mathbb{P}_\varepsilon \bar{\Delta} \mathbb{P}_\varepsilon = 0$. The following proposition describes the term of order 2 in star-products.

Proposition 6.2. *If $\mathbb{P} = \sum_{i,j=0}^{n} a_{ij} D_i \otimes D_j \in \Lambda^2 \mathscr{L}$, $a_{ij} = -a_{ji}$, is a Poisson element, then $\mathbb{P}\Delta\mathbb{P} = \partial_{\mathbb{1}\otimes\mathbb{1}} A_2$, where*

$$A_2 = \tfrac{1}{2}\mathbb{P}^2 + \frac{1}{6}\sum_{i,j,k,l} a_{ij}a_{kl}(D_i[D_l, D_j]\otimes D_k + D_k\otimes D_i[D_l, D_j]) + \partial_{\mathbb{1}\otimes\mathbb{1}} B\,,$$

and $B \in \mathscr{U}$.

The proof is just a matter of computations and we omit it. Note only that $\mathbb{P}^2 = \mathbb{P}\cdot\mathbb{P}$ is the square of $\mathbb{P}$ in the algebra $\mathscr{U}\otimes\mathscr{U}$.

Let us now see which problem appears when we try to obtain the higher-order deformations inductively. Suppose that $\mathbb{1}\otimes\mathbb{1} + \varepsilon\mathbb{P} + \varepsilon^2 A_2$ is a star-product of order 2. Thus $\mathbb{P}\Delta A_2$ is a $\partial_{\mathbb{1}\otimes\mathbb{1}}$ cocycle. Define $\sigma : \mathscr{U}^3 \to \mathscr{U}^3$ by $\sigma(x\otimes y\otimes z) = z\otimes y\otimes x$. It is easy to see that $\sigma(\partial_{\mathbb{1}\otimes\mathbb{1}} A) = \partial_{\mathbb{1}\otimes\mathbb{1}} A$ if $A \in \mathscr{U}^2$ is antisymmetric and $\sigma(\partial_{\mathbb{1}\otimes\mathbb{1}} A) = -\partial_{\mathbb{1}\otimes\mathbb{1}} A$ if A is symmetric. One can check that $\sigma(\mathbb{P}\Delta A_2) = \mathbb{P}\Delta A_2$, so that the antisymmetric part of $\mathbb{P}\Delta A_2$ vanishes and $\mathbb{P}\Delta A_2 = \partial_{\mathbb{1}\otimes\mathbb{1}} A_3$. Moreover, by the above property of σ, A_3 can be chosen antisymmetric. Since $\mathbb{P}$ and A_2 contain no elements of the form $\mathbb{1}\otimes X$, $\mathbb{P}\Delta A_2$ contains no elements of the form $\mathbb{1}\otimes X\otimes Y$ or $X\otimes\mathbb{1}\otimes Y$, and A_3 can be chosen "vanishing on constants" as well. Unfortunately, direct computations of A_3 are complicated and we omit them.

To obtain the term of order 4, it is now necessary that $2\mathbb{P}\Delta A_3 + A_2\Delta A_2$ is a coboundary. Since it is a cocycle, it is a coboundary if and only if its antisymmetric part vanishes, i.e., if and only if $\mathbb{P}\bar{\Delta} A_3 = 0$ and we must work in "Chevalley cohomology". This is the most important step in the symplectic case and it is overcome by means of hard computations using properties of symplectic connections.

It is interesting that the whole procedure can be repeated in an exponential notation. Namely, we shall look for star-products of the form $\mathbb{A}_\varepsilon = \exp(\varepsilon\mathbb{P} + \varepsilon^2 B_2 + \cdots)$, where the exponential is taken with respect to the associative algebra structure of $\mathscr{U}^2_\varepsilon$. Since $c_0, c_1, I_0, I_2 : \mathscr{U}^2 \to \mathscr{U}^3$ are homomorphisms of associative algebras, the condition $\mathbb{A}_\varepsilon \Delta \mathbb{A}_\varepsilon = 0$ can be written in the form

$$\begin{aligned}&\exp(\varepsilon c_0(\mathbb{P}) + \varepsilon^2 c_0(B_2) + \cdots)\cdot\exp(\varepsilon\mathbb{P}_2 + \varepsilon^2 I_2(B_2) + \cdots)\\ &\quad = \exp(\varepsilon c_1(\mathbb{P}) + \varepsilon^2 c_1(B_2) + \cdots)\cdot\exp(\varepsilon\mathbb{P}_0 + \varepsilon^2 I_0(B_2) + \cdots)\,. \qquad (6.1)\end{aligned}$$

We find the condition for B_2 using the Campbell–Hausdorff formula and the fact that exponents coincide up to order k if and only if the arguments coincide up to order k. Thus we get from (6.1)

$$\tfrac{1}{2}[c_0(\mathbb{P}),\mathbb{P}_2]+c_0(B_2)+I_2(B_2)=\tfrac{1}{2}[c_1(\mathbb{P}),\mathbb{P}_0]+c_1(B_2)+I_0(B_2)\,,$$

where $[\cdot,\cdot]$ is the commutator in $\mathscr{U}^3$. Thus

$$\partial_{\mathbb{1}\otimes\mathbb{1}}B_2=\tfrac{1}{2}([c_1(\mathbb{P}),\mathbb{P}_0]-[c_0(\mathbb{P}),\mathbb{P}_2])\,,$$

and since, as one can easily check, $c_0(\mathbb{P})=\mathbb{P}_0+\mathbb{P}_1$, $c_1(\mathbb{P})=\mathbb{P}_1+\mathbb{P}_2$, and $\mathbb{P}\bar{\Delta}\mathbb{P}=([\mathbb{P}_0,\mathbb{P}_1]+[\mathbb{P}_0,\mathbb{P}_2]+[\mathbb{P}_1,\mathbb{P}_2])=0$, we get finally

$$\partial_{\mathbb{1}\otimes\mathbb{1}}B_2=\tfrac{1}{2}[\mathbb{P}_2,\mathbb{P}_0]\,.$$

It is not hard to verify that B_2 satisfies this equation if and only if it is of the form $A_2-\frac{1}{2}\mathbb{P}^2$ for A_2 as in proposition 6.2. This is not surprising, since the term $\frac{1}{2}\mathbb{P}^2$ is included in the exponential notation.

Further calculations become complicated and we note only that we have to look for $B_3\in\mathscr{U}^2$ satisfying

$$\begin{aligned}2\partial_{\mathbb{1}\otimes\mathbb{1}}B_3=&[\mathbb{P}_1+\mathbb{P}_2,c_0(B_2)]+[c_1(B_2),\mathbb{P}_0+\mathbb{P}_1]+[\mathbb{P}_2,I_0(B_2)]\\&+[I_2(B_2),\mathbb{P}_0]+\tfrac{1}{6}[\mathbb{P}_2+\mathbb{P}_0,[\mathbb{P}_2,\mathbb{P}_0]]\,.\end{aligned}$$

There is a case where we can give a simple answer. In this case all terms of higher orders vanish.

Theorem 6.3. *If $\mathbb{P}=\sum_{i,j}a_{ij}D_i\otimes D_j\in\Lambda^2\mathscr{L}$ is a Poisson element such that $[\mathbb{P}_2,\mathbb{P}_0]=0$ (i.e., the D_i commute pairwise), then $\mathbb{P}$ is a Poisson element and* $\exp(\varepsilon\mathbb{P})=\sum_{k=0}^{\infty}\varepsilon^k\mathbb{P}^k/k!$ *is a star-product for $\mathbb{P}$.*

Proof. By permutation of variables, $[\mathbb{P}_0,\mathbb{P}_2]=0$ implies $[\mathbb{P}_1,\mathbb{P}_2]=0$ and $[\mathbb{P}_0,\mathbb{P}_1]=0$ and hence the Yang–Baxter equation $\mathbb{P}\bar{\Delta}\mathbb{P}=0$, so $\mathbb{P}$ is a Poisson element. Put $\mathbb{A}_\varepsilon=\exp(\varepsilon\mathbb{P})$. Then both sides of (6.1) are equal to $\exp[\varepsilon(\mathbb{P}_0+\mathbb{P}_1+\mathbb{P}_2)]$, so $\mathbb{A}_\varepsilon\Delta\mathbb{A}_\varepsilon=0$. □

However, there is a general formula which can be derived from Drinfel'd [12]. Let us sketch briefly the idea.

A Poisson element $\mathbb{P}=\sum_{i,j}\alpha_{ij}D_i\otimes D_j\in\Lambda^2\mathscr{L}$ can be regarded as a left-invariant Poisson structure on the corresponding Lie group G. After reduction to the symplectic leaf through the neutral element e which is a Lie subgroup, we can assume that $\det(\alpha_{ij})\neq 0$, i.e., that $\mathbb{P}$ corresponds to the left-invariant symplectic form Ω on G. As a form on $\mathscr{L}$ regarded as the space of left-invariant vector fields, $\Omega=$

$(b_{ij})=(\alpha_{ij})^{-1}$ in the basis $\{D_i\}$. Since Ω is closed, it defines a cocycle, i.e., a central extension $\hat{\mathscr{L}}=\mathscr{L}\oplus\mathbb{R}\hbar$ of the Lie algebra $\mathscr{L}$. The Lie bracket $[\ ,\ ]\hat{}$ in $\hat{\mathscr{L}}$ is defined for elements of $\mathscr{L}\subseteq\hat{\mathscr{L}}$ by $[D_i,D_j]\hat{}=[D_i,D_j]+b_{ij}\hbar$. The Lie–Poisson structure on the dual $(\hat{\mathscr{L}})^*$ [with the symplectic leaves being affine subspaces of $(\hat{\mathscr{L}})^*$] admits the star-product $\hat{\mathbb{A}}_\varepsilon$ described in theorem 4.6. This product is invariant with respect to the coadjoint action of G. Take $\varphi\in(\hat{\mathscr{L}})^*$, $\varphi(\hbar)=1$, $\varphi(\mathscr{L})=0$. The coadjoint action of G in $(\hat{\mathscr{L}})^*$ gives us a local diffeomorphism $\Phi: G\to\mathcal{O}(\hbar)$ of G onto the orbit $\mathcal{O}(\hbar)$ of $\hbar$. This mapping is equivariant, so the pullback $\mathbb{A}_\varepsilon=\Phi_*^{-1}(\hat{\mathbb{A}}_\varepsilon)$ is a star-product for the symplectic structure on G given by left-invariant differential operators. The algebra of left-invariant differential operators is canonically isomorphic to the universal enveloping algebra $\mathscr{U}(\mathscr{L})$, i.e., we get a star-product for $\mathbb{P}$. Thus we get the following theorem.

Theorem 6.4 (Drinfel'd). *Every Poisson element*

$$\mathbb{P}=\sum_{i,j}\alpha_{ij}D_i\otimes D_j\in\Lambda^2\mathscr{L}$$

admits a star-product $\mathbb{A}_\varepsilon\in\mathscr{U}_\varepsilon^{\otimes 2}$.

In spite of the fact that the above procedure is constructive, the explicit form of $\mathbb{A}_\varepsilon$ is rather complicated. Every linear differential operator on G can be written in the form $\sum_\alpha f_\alpha D^\alpha$ with $f_\alpha\in C^\infty(G)$ and $D^\alpha=D_1^{\alpha_1}\circ\cdots\circ D_n^{\alpha_n}\in\mathscr{U}(\mathscr{L})$. Having the form of $\hat{\mathbb{A}}_\varepsilon$ as in theorem 4.6, we can write

$$\mathbb{A}_\varepsilon=\mathbb{1}\otimes\mathbb{1}+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{|\alpha_i|+|\beta_i|>1}\varphi(\hat{c}_{\alpha_1\beta_1})\cdots\varphi(\hat{c}_{\alpha_n\beta_n})(2\varepsilon)^{\sum(|\alpha_i|+|\beta_i|)-n}$$
$$\times\Phi_*^{-1}(\partial^{\sum\alpha_i})(e)\otimes\Phi_*^{-1}(\partial^{\sum\beta_i})(e)\,,$$

where $\hat{c}_{\alpha\beta}$ are the coefficients in the Campbell–Baker–Hausdorff series,

$$\widehat{\mathrm{CH}}(\sum t_kD_k,\sum s_jD_j)=\sum t_kD_k+\sum s_jD_j+\tfrac{1}{2}\varepsilon\sum t_ks_j[D_k,D_j]\hat{}+\cdots,$$

of the term $t^\alpha s^\beta$, and

$$(\sum f_\alpha D^\alpha)(e)=\sum f_\alpha(e)D^\alpha\in\mathscr{U}(\mathscr{L})\,.$$

Since

$$\Phi_*^{-1}(\partial^\alpha)=\Phi_*^{-1}(\partial_1)^{\alpha_1}\circ\cdots\circ\Phi_*^{-1}(\partial_n)^{\alpha_n}\,,$$

we can calculate $\Phi_*^{-1}(\partial^\alpha)(e)$ using the formulae

$$\Phi_*^{-1}(\partial_m)=\sum h_{mi}\alpha_{ni}D_i,\qquad D_k(h_{mn})=\sum h_{mi}c^i_{kn}\,,$$

where h_{mn} are the matrix elements of the adjoint action of G in the basis $\{D_i\}$ and c^i_{kn} are the structure constants. Since clearly $h_{mn}(e)=\delta_{mn}$, we get (cf. proposition 6.2 and ref. [12])

$$\begin{aligned}\mathbb{A}_\varepsilon &= \mathbb{1}\otimes\mathbb{1}+\varepsilon\mathbb{P}\\ &+\varepsilon^2\left(\frac{1}{2}\sum \alpha_{na}\alpha_{mb}D_mD_n\otimes D_aD_b\right.\\ &+\frac{1}{6}\sum \alpha_{ni}\alpha_{ja}c^m_{ij}(D_mD_n\otimes D_a+D_a\otimes D_mD_n)\\ &\left.+\frac{1}{6}\sum \alpha_{mk}\alpha_{aj}c^l_{ij}c^i_{lk}D_m\otimes D_a\right)+o(\varepsilon^2)\,. \end{aligned}\tag{6.2}$$

Note that we can get a star-product for a Poisson element $\mathbb{P}=\sum_{i,j}\alpha_{ij}D_i\otimes D_j\in\Lambda^2\mathscr{L}$ for $\mathscr{L}$ being a Lie algebra over an arbitrary field of characteristic zero in a similar way.

The above theorem enables us to construct star-products for Poisson structures defined by Poisson elements. Let (V,P) be a Poisson algebra with a Poisson structure P of the form $P=\sum_{i,j=1}^n a_{ij}D_i\cdot D_j$, where $D_i\in\mathrm{Der}(V)$ and $a_{ij}\in\mathscr{k}$, $a_{ij}=-a_{ji}$, j, $i=1, ..., n$. Let $\mathscr{L}$ be the Lie subalgebra in $\mathrm{Der}(V)$ generated by $\{D_i\}$ and let $\mathscr{U}$ be the universal enveloping algebra over $\mathscr{L}$. The inclusion $J:\mathscr{L}\to\mathrm{Der}(V)$ extends to a homomorphism $J:\mathscr{U}\to\mathrm{Diff}(V)$ of associative algebras and further to a homomorphism $J:(\mathscr{U}^\otimes,\Delta)\to(\mathrm{Diff}^*(V),\Delta)$ of the graded Lie algebras as in theorem 5.4.

If $\mathbb{P}=\sum_{i,j=1}^n a_{ij}D_i\otimes D_j$ is a Poisson element in $\mathscr{U}^\otimes$, then it has a star-product $\mathbb{A}_\varepsilon$ and hence $A_\varepsilon=J(\mathbb{A}_\varepsilon)$ is a star-product for P.

Corollary 6.5. *Let $P=\sum_{i,j=1}^n a_{ij}D_i\cdot D_j$, where $D_i\in\mathrm{Der}(V)$ and $a_{ij}\in\mathscr{k}$, $a_{ij}=-a_{ji}$, i, $j=i, ..., n$, be a Poisson structure on an associative commutative algebra (V,m) over a field $\mathscr{k}$ of characteristic 0. If $\mathbb{P}=\sum_{i,j=1}^n a_{ij}D_i\otimes D_j\in\mathscr{U}\otimes_{\mathscr{k}}\mathscr{U}$ is a Poisson element, then P has a star-product of the form $J(\mathbb{A}_\varepsilon)$, where $\mathbb{A}_\varepsilon$ is the star-product of $\mathbb{P}$ as in* (6.2).

Remark 6.6. We usually have some freedom in writing a given Poisson structure in the form $P=\sum_{i,j}a_{ij}D_i\cdot D_j$, with $a_{ij}\in\mathscr{k}$ and $D_i\in\mathrm{Der}(V)$. Even if $\mathrm{Der}(V)$ is locally freely generated as a V module, we have for a given set of generators a unique such form, but with a_{ij} from V. Since we must consider tensor products over $\mathscr{k}$ and not over V in order to have an associative algebra structure, we can, for instance, decide whether to write the element $u(D_1\otimes D_2)$ with $u\in V$, as $(uD_1)\otimes D_2$ or $D_1\otimes(uD_2)$, etc.

This makes our approach far from unique, but it seems that it lies in the nature of the problem. On the other hand, there is always the hope that we can choose $\mathbb{P}$

for P in such a way that $\mathbb{P}$ is a Poisson element and get the star-product in the form $J(\mathbb{A}_\varepsilon)$.

If we are unable to write a corresponding Poisson element we always have a star-product of order 2 and hence, by the cohomology properties, of order 3. The question of the existence of a full star-product for a given Poisson structure is open.

Theorem 6.7. *Let $P=\sum_{i,j=1}^{n} a_{ij}D_i\cdot D_j$, $D_i\in \mathrm{Der}(V)$ and $a_{ij}\in \mathscr{k}$, $a_{ij}=-a_{ji}$, $i,j=1,\dots,n$, be a Poisson structure on associative commutative algebra (V,m). Then P has a star-product $m+\varepsilon P+\varepsilon^2A_2+\varepsilon^3A_3$ of order 3. The term A_2 can be written as*

$$A_2(u,v)=\frac{1}{2}\sum_{i,j,k,l} a_{ij}a_{kl}D_iD_k(u)D_jD_l(v)$$
$$+\frac{1}{6}\sum_{i,j,k,l} a_{ij}a_{kl}(D_i[D_l,D_j](u)D_k(v)+D_k(u)D_i[D_l,D_j](v)) .$$

Proof. This is just a matter of direct computations. □

We shall end with some examples of Poisson structures and corresponding star-products.

Example 6.8. We always have a star-product for a symplectic structure locally, since the corresponding Poisson structure can be written in suitable coordinates $(x_1,\dots,x_{2n})$ in the form $P=2\sum_{i=1}^{n}\partial_i\wedge\partial_{i+n}$, where ∂_i stands for $\partial/\partial x_i$, and derivations $\{\partial_i\}$ commute. The form of P is invariant with respect to a suitable change of coordinates, but the derivations ∂_i do not change linearly (over $\mathbb{R}$), i.e., $\mathbb{P}=\sum_{i=1}^{n}(\partial_i\otimes\partial_{i+n}-\partial_{i+n}\otimes\partial_i)$ is not preserved as an element of the tensor product over $\mathbb{R}$ and $\exp(\varepsilon\mathbb{P})$ does not define a star-product globally. In special cases only, e.g. when we have a torsion-free symplectic connection without curvature, does a suitable change of coordinates preserve $\mathbb{P}$ and do we get the Moyal star-product.

Example 6.9. Consider the Poisson structure on $\mathbb{R}^4$ defined on coordinate functions $a, b, x, y\in C^\infty(\mathbb{R}^4)$ by $P(y,a)=yb$, $P(b,y)=ya$, $P(b,x)=xa$, $P(x,a)=xb$, $P(x,y)=P(a,b)=0$. Putting $A=x\partial_x+y\partial_y$, $B=b\partial_a-a\partial_b$, one can write P in the form $P=2A\wedge B$. This is a quadratic Poisson structure. Since A and B commute, we have a star-product $\sum_{k=0}^{\infty}\varepsilon^kP^k/k!$, where

$$P^k(u,v)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}A^{k-i}B^i(u)A^iB^{k-i}(v) .$$

Example 6.10. Consider an interesting Poisson structure P on the unit sphere $S^2 \subset \mathbb{R}^3$ defined in global coordinates (x,y,z) in $\mathbb{R}^3$ by

$$P(x,y)=(1-z)^2 z, \quad P(z,x)=(1-z)^2 y, \quad P(y,z)=(1-z)^2 x,$$

so that the corresponding two-tensor field P can be written in the form

$$P=2(1-z)^2(zY\wedge X+xY\wedge Z+yX\wedge Z),$$

where $X=z\partial_y-y\partial_z$, $Y=z\partial_x-x\partial_z$, $Z=x\partial_y-y\partial_x$ are vector fields on S^2. P is degenerate at $(0,0,1)$ and the rest of S^2 forms one two-dimensional symplectic leaf of the corresponding Kirillov foliation. The vector fields X, Y, Z form the Lie algebra $\mathrm{sl}(2,\mathbb{R})$, but at some effort one can write P using commuting vector fields only. Namely, $P=J(\mathbb{P})$, where $\mathbb{P}=A\otimes B-B\otimes A$ and

$$A=-xy\partial_x+(1-z-y^2)\partial_y+(1-z)y\partial_z,$$
$$B=(1-z-x^2)\partial_x-xy\partial_y+(1-z)x\partial_z,$$

are tangent to S^2 and commuting vector fields. Thus we get a star-product for P in the form formally as in example 6.9. Of course the coordinate form of this star-product is rather complicated.

Example 6.11. Let the group $G=\mathbb{Z}_2$ act on $\mathbb{R}^2$ by the reflection $(x,y)\to(-x,y)$. The orbit space $N=\mathbb{R}^2/G$ is a closed half-plane with the differentiable structure induced from $\mathbb{R}^2$, i.e.,

$$V=C^\infty(N)=\{f\in C^\infty(\mathbb{R}^2) : f(x,y)=f(-x,y)\}.$$

The vector fields $A=\partial_y$ and $B=(1/x)\partial_x$ define commuting derivations of V (the limit $\lim_{x\to 0} B(f)(x,y)$ exists for every $f\in V$) and hence $P=A\wedge B$ is a Poisson structure on N. A star-product for P is formally as in example 6.9.

We are most grateful for the extraordinary hospitality and mathematical atmosphere provided by Professor R. Nagel and his colleagues at the University of Tübingen. We would also like to thank Professor C. Roger, who turned our attention to ref. [12].

References

[1] C.J. Atkin, A note on the algebra of Poisson brackets, Math. Proc. Camb. Philos. Soc. 96 (1984) 45–60.
[2] C.J. Atkin and J. Grabowski, Homomorphisms of the Lie algebras associated with a symplectic manifold, Compos. Math. 76 (1990) 315–349.
[3] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Deformation theory and quantization, Ann. Phys. 111 (1978) 61–110, 111–151.
[4] K.H. Bhaskara and K. Viswanath, *Poisson Algebras and Poisson Manifolds* (Longman, London, 1988).

[5] N. Bourbaki, *Groupes et algèbres de Lie* (Hermann, Paris, 1972) Ch. II, § 11.
[6] J. Braconnier, Eléments d'algèbre différentielle gradueé, Publ. Dep. Math., Univ. C. Bernard, Lyon 1 (1982) 1/C.
[7] M. Cahen, M. De Wilde and S. Gutt, Local cohomology of the algebra of C^∞ functions on a connected manifold, Lett. Math. Phys. 4 (1980) 157–167.
[8] A. Connes, Non commutative differential geometry, Publ. Math. IHES 62 (1985) 41–144.
[9] M. De Wilde and P.B.A. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. 7 (1983) 487–496.
[10] M. De Wilde and P.B.A. Lecomte, Formal deformations of the Poisson algebra of a symplectic manifold and star-products. Existence, equivalence, derivations, in: *Deformation Theory of Algebras and Structures and Applications*, eds. M. Hazewinkel and M. Gerstenhaber, NATO ASI Ser. C., Vol. 247 (Kluwer Acad. Publ., 1988) pp. 897–960.
[11] J. Dixmier, *Enveloping Algebras* (Akademie-Verlag, Berlin, 1977) pp. 89–99.
[12] W.G. Drinfel'd, About constant quasiclassical solutions of the quantum Yang–Baxter equations, Dokl. Akad. Nauk SSSR 273 (1983) 531–525 (in Russian).
[13] M. Gerstenhaber, On the deformations of rings and algebras, Ann. Math. 79 (1964) 59–103; 84 (1966) 1–12; 88 (1968) 1–34; 99 (1974) 257–276.
[14] J. Grabowski, The Lie structure of C* and Poisson algebras, Studia Math. 81 (1985) 259–270.
[15] F. Guedira and A. Lichnerowicz, Géométrie des algèbres de Lie locales de Kirillov, J. Math. Pures Appl. 63 (1984) 407–484.
[16] V. Guillemin and S. Sternberg, The moment map and collective motion, Ann. Phys. 127 (1980) 220–253.
[17] S. Gutt, An explicit $*$-product on the cotangent bundle of a Lie group, Lett. Math. Phys. 7 (1983) 249–258.
[18] A.A. Kirillov, Local Lie algebras, Usp. Mat. Nauk 31 (1976) 57–76 [Russ. Math. Surv. 31 (1976) No. 4, 55–75].
[19] I.S. Krasil'shchik, The Schouten brackets and canonical algebras, Lect. Notes Math. 1334 (1988) 79–110.
[20] I.S. Krasil'shchik, Super-canonical algebras and Schouten brackets, Mat. Zam. 49 (1991) No. 1, 70–76 (in Russian).
[21] I.S. Krasil'shchik and A.M. Vinogradov, What is the Hamiltonian formalism?, Usp. Mat. Nauk 30 (1975) No. 1, 173–198 (in Russian).
[22] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, J. Diff. Geom. 12 (1977) 253–300.
[23] A. Lichnerowicz, Les variétés de Jacobi et leurs algèbres de Lie associées, J. Math. Pures Appl. 57 (1978) 453–488.
[24] J.E. Moyal, Quantum mechanics as a statistical theory, Proc. Camb. Philos. Soc. 45 (1949) 99–124.
[25] O.M. Neroslavsky and A.T. Vlassov, Sur les déformations de l'algèbre des fonctions d'une variété symplectique, C.R. Acad. Sci. Paris Ser. I 292 (1981) 71–73.
[26] A. Nijenhuis and R. Richardson, Deformation of Lie algebra structures, J. Math. Mech. 17 (1967) 89–105.
[27] J.A. Schouten, Über Differentialkonkomitanten zweier kontravarianter Grössen, Indag. Mat. 2 (1940) 449–452.
[28] J. Vey, Déformations du crochet de Poisson d'une variété symplectique, Commun. Math. Helv. 50 (1975) 421–454.
[29] A.M. Vinogradov, The logic algebra for the theory of linear differential operators, Sov. Math. Dokl. 13 (1972) 1058–1062.
[30] A. Weinstein, The local structure of Poisson manifolds, J. Diff. Geom. 18 (1983) 523–557.